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The Casimir effect in the boundary state formalism

Z Bajnok[1], **L Palla**[2] **and G Takács**[1]

[1] Theoretical Physics Research Group, Hungarian Academy of Sciences, 1117 Budapest, Pázmány Péter sétány 1/A, Hungary
[2] Institute for Theoretical Physics, Eötvös University, 1117 Budapest, Pázmány Péter sétány 1/A, Hungary

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Abstract
The Casimir effect in the planar setting is described using the boundary state formalism, for general partially reflecting boundaries. It is expressed in terms of the low-energy degrees of freedom, which provides a large distance expansion valid for general interacting field theories provided there is a non-vanishing mass gap. The expansion is written in terms of the scattering amplitudes, and needs no ultraviolet renormalization. We also discuss the case when the quantum field has a nontrivial vacuum configuration.

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1. Introduction

The Casimir effect can be considered as a response of the ground state in a quantum field theory to the presence of boundary conditions. Therefore it is natural to seek a relation to the approach known as boundary quantum field theory started in two-dimensional spacetime by the seminal paper of Ghoshal and Zamolodchikov [1]. Recently we have developed and extended this formalism to quantum field theories in arbitrary spacetime dimensions and applied it to the Casimir effect [2–5]. Here we give a short review of our results.

2. Boundary state formalism

2.1. The concept of the boundary state

Following [5] we consider an Euclidean quantum field theory of a scalar field Φ defined in a $(D+1)$-dimensional half spacetime, parameterized as $(x \leqslant 0, y, \vec{r})$, in the presence of a codimension one flat boundary located at $x = 0$. The correlation functions defined as

$$\langle \Phi(x_1, y_1, \vec{r}_1) \cdots \Phi(x_N, y_N, \vec{r}_N) \rangle = \frac{\int \mathcal{D}\Phi \, \Phi(x_1, y_1, \vec{r}_1) \cdots \Phi(x_N, y_N, \vec{r}_N) \, e^{-S[\Phi]}}{\int \mathcal{D}\Phi \, e^{-S[\Phi]}}$$

contain all information about the theory. The measure in the functional integral is provided by the classical action

$$S[\Phi] = \int \mathrm{d}\vec{r} \int_{-\infty}^{\infty} \mathrm{d}y \left[\int_{-\infty}^{0} \mathrm{d}x \left(\frac{1}{2}(\vec{\nabla}\Phi)^2 + U(\Phi) \right) + U_B(\Phi(x=0, y, \vec{r})) \right],$$

which determines also the boundary condition via the boundary potential U_B,

$$\partial_x \Phi|_{x=0} = -\frac{\delta U_B(\Phi)}{\delta \Phi}\bigg|_{x=0}.$$

Here we assume for simplicity that the boundary term does not depend on the temporal (i.e. y) derivative of Φ, which means that there are no boundary degrees of freedom with a temporal dynamics independent of the bulk (it may depend on derivatives with respect to $\vec{r}$, which is the reason for the variational derivative δ). The bulk interaction U is constrained by the requirement that the bulk spectrum must possess a mass gap m.

This Euclidean quantum field theory can be considered as the imaginary time version of two different Minkowskian quantum field theories. We can consider $t = -\mathrm{i}y$ as Minkowskian time and so the boundary is located in space providing a nontrivial boundary condition for the field Φ. The space of states in this Hamiltonian description is the boundary Hilbert space $\mathcal{H}_B$ determined by the configurations on the equal time slices. $\mathcal{H}_B$ contains multi-particle states and is built over the vacuum state, obtained in the presence of the boundary condition ($|0\rangle_B$), by the successive application of particle creation operators[3]. In the asymptotic past the particles do not interact and behave as free particles travelling towards the boundary; thus

$$\mathcal{H}_B = \left\{ a_{\text{in}}^{+}(k_1, \vec{k}_1) \cdots a_{\text{in}}^{+}(k_N, \vec{k}_N)|0\rangle_B, k_1 \geqslant \cdots \geqslant k_N > 0 \right\},$$

where the operator $a_{\text{in}}^{+}(k, \vec{k})$ creates an asymptotic particle of mass m with transverse (i.e. x-directional) momentum k and parallel (i.e. parallel to the boundary) momentum $\vec{k}$. The corresponding energy is $\omega(k, \vec{k}) = \sqrt{k^2 + \vec{k}^2 + m^2} = \sqrt{k^2 + m_{\text{eff}}(\vec{k})^2}$ where $m_{\text{eff}}(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$ is the effective mass of a particle with parallel momentum $\vec{k}$ as seen in the two-dimensional spacetime formed by t and x. Instead of k, we shall also frequently use the rapidity parameter ϑ defined by

$$\omega = m_{\text{eff}}(\vec{k}) \cosh \vartheta, \qquad k = m_{\text{eff}}(\vec{k}) \sinh \vartheta. \tag{2.1}$$

In the Heisenberg picture the time evolution of the field

$$\Phi(x, t, \vec{r}) = \mathrm{e}^{\mathrm{i}H_B t}\, \Phi(x, 0, \vec{r})\, \mathrm{e}^{-\mathrm{i}H_B t}$$

is generated by the following boundary Hamiltonian:

$$H_B = \int \mathrm{d}\vec{r} \left[\int_{-\infty}^{0} \mathrm{d}x \left(\frac{1}{2}\Pi_t^2 + \frac{1}{2}(\partial_x \Phi)^2 + \frac{1}{2}(\vec{\partial}\Phi)^2 + U(\Phi) \right) + U_B(\Phi(x=0)) \right]. \tag{2.2}$$

The correlator can then be understood as the matrix element

$$\langle \Phi(x_1, y_1, \vec{r}_1) \cdots \Phi(x_N, y_N, \vec{r}_N) \rangle =_B \langle 0| T_t\left(\Phi(x_1, t_1, \vec{r}_1) \cdots \Phi(x_N, t_N, \vec{r}_N) \right) |0\rangle_B,$$

where T_t denotes time ordering with respect to time t, and the vacuum $|0\rangle_B$ is normalized to 1.

The formulation of asymptotic states and fields, together with the relevant reduction formulae (which generalize the LSZ approach to boundary QFT) was given in [2, 3]. In [3] we also gave the appropriate generalization of Landau equations, Coleman–Norton interpretation and Cutkosky rules, together with an example of one-loop renormalization of boundary interaction (where we considered the case of sine-Gordon model in two spacetime dimensions).

[3] One can also introduce particle-like excitations confined to the boundary [3] ('surface plasmons'), but for simplicity we do not consider them here.

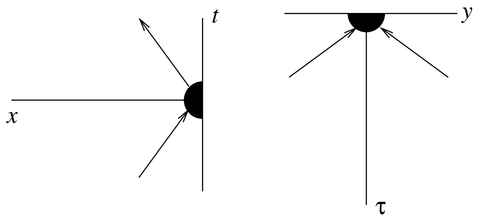

Figure 1. The two Hamiltonian descriptions, with a representation of the amplitudes R and K^2.

Elastic reflection of a particle from the boundary (see figure 1) is a process with one particle of energy ω and parallel momentum $\vec{k}$ both in the incoming and outgoing state[4], whose transverse momentum k changes sign. Its amplitude is the *reflection factor* $R(\omega, \vec{k})$ which can only depend on $|\vec{k}|$, as a result of rotational invariance in the directions parallel to the boundary; it is not necessarily unitary due to the possible existence of inelastic processes.

Alternatively we can also consider $\tau = -\mathrm{i}x$ as Minkowskian time, as depicted in figure 1. In this case, the boundary is located in time and we can use the usual infinite volume Hamiltonian description. The Hilbert space is the bulk Hilbert space $\mathcal{H}$ spanned by multi-particle *in* states

$$\mathcal{H} = \left\{A^+_{\mathrm{in}}(\kappa_1, \vec{k}_1) \cdots A^+_{\mathrm{in}}(\kappa_N, \vec{k}_N)|0\rangle, k_1 \geqslant \cdots \geqslant k_N\right\},$$

where κ is the momentum in the y direction, and the energy corresponding to the time direction is given by $\omega(\kappa, \vec{k}) = \sqrt{m^2 + \kappa^2 + \vec{k}^2}$. One can again use a rapidity parametrization in this channel defined by

$$\kappa = m_{\mathrm{eff}}(\vec{k}) \sinh\vartheta, \qquad \omega = m_{\mathrm{eff}}(\vec{k}) \cosh\vartheta \tag{2.3}$$

Time evolution

$$\Phi(\tau, y, \vec{r}) = \mathrm{e}^{\mathrm{i}H\tau}\, \Phi(0, y, \vec{r})\, \mathrm{e}^{-\mathrm{i}H\tau}$$

is generated by the bulk Hamiltonian

$$H = \int \mathrm{d}\vec{r} \int_{-\infty}^{\infty} \mathrm{d}y \left(\frac{1}{2}\Pi_\tau^2 + \frac{1}{2}(\partial_y\Phi)^2 + \frac{1}{2}(\vec{\partial}\Phi)^2 + U(\Phi)\right) \tag{2.4}$$

and the boundary appears in time as a final state in calculating the correlator,

$$\langle\Phi(x_1, y_1, \vec{r}_1) \cdots \Phi(x_N, y_N, \vec{r}_N)\rangle = \langle B|T_\tau\,(\Phi(\tau_1, y_1, \vec{r}_1) \cdots \Phi(\tau_N, y_N, \vec{r}_N))\,|0\rangle.$$

The state $\langle B|$ is called the boundary state, which is an element of the bulk Hilbert space and is defined by the equality of the two alternative Hamiltonian descriptions

$$\langle B|T_\tau\,(\Phi(\tau_1, y_1, \vec{r}_1) \cdots \Phi(\tau_N, y_N, \vec{r}_N))\,|0\rangle =_B \langle 0|T_t\,(\Phi(x_1, t_1, \vec{r}_1) \cdots \Phi(x_N, t_N, \vec{r}_N))\,|0\rangle_B,$$

where the correspondence is valid if $(\mathrm{i}\tau, y)$ is identified with $(x, \mathrm{i}t)$. Using asymptotic completeness the boundary state can be expanded in the basis of asymptotic *in* states as[5]

$$\langle B| = \langle 0|\Bigg\{1 + \bar{K}^1 A_{\mathrm{in}}(0, 0) + \int_0^\infty \frac{\mathrm{d}\kappa}{2\pi} \int \frac{\mathrm{d}^{D-1}\vec{k}}{(2\pi)^{D-1}\omega(\kappa, \vec{k})} \bar{K}^2(\kappa, \vec{k}) A_{\mathrm{in}}(-\kappa, -\vec{k}) A_{\mathrm{in}}(\kappa, \vec{k}) + \cdots\Bigg\}, \tag{2.5}$$

[4] Energy and parallel momentum are conserved due to the unbroken translation invariance in the directions parallel to the boundary.

[5] The bars on top of the K coefficients indicate that the above expansion is that of the conjugate ('bra') boundary state.

which we refer to as the cluster expansion for the boundary state (where due to translational invariance only bulk multi-particle states with zero total momentum can appear).

2.2. *Relation between the two channels: K^1 and K^2 in terms of R*

The one-point function of the field, due to unbroken Poincaré symmetry along the boundary, only has a nontrivial dependence on x,

$$_B\langle 0|\Phi(x,t,\vec{r})|0\rangle_B = G^1_{\mathrm{bdry}}(x),$$

which corresponds to a nontrivial vacuum configuration in the presence of the boundary condition. The leading asymptotic behaviour for $x \to -\infty$ is given by [5]

$$_B\langle 0|\Phi(x,t,\vec{r})|0\rangle_B = \langle 0|\Phi(0)|0\rangle + \bar{g}\,\mathrm{e}^{mx}, \tag{2.6}$$

where $\langle 0|\Phi(0)|0\rangle$ is the vacuum expectation value in the bulk and $\bar{g}$ is a parameter which is characteristic of the boundary condition (and also of the field Φ). We recall that $|0\rangle_B$ is the ground state of the boundary system which means that there are no bulk excitations present and the boundary itself is in its ground state. The absence of bulk excitations is important for the above asymptotics to be valid; however, (2.6) also holds when the boundary is excited ('surface plasmons').

Using the property of the interpolating field Φ

$$\langle 0|\Phi(0)|A(\underline{k}=\underline{0})\rangle = \sqrt{\frac{Z}{2}},$$

where Z is the bulk wavefunction renormalization constant ($0 \leqslant Z < 1$), and from the cluster expansion (2.5) one obtains the relation[6]

$$\bar{g} = \sqrt{\frac{Z}{2}}\bar{K}^1.$$

On the other hand, the existence of nontrivial vacuum expectation value for the field is generally related to a singularity of the reflection factor at the particular kinematical point $\vec{k}=0$, $\omega=0$ (i.e. $k = im$ or equivalently $\vartheta = \mathrm{i}\pi/2$). In our paper [5] it was shown that this singularity takes the following form:

$$R(\omega,\vec{k}) \sim -\frac{mg^2/2}{\omega}(2\pi)^D\delta(\vec{k}), \tag{2.7}$$

with g parametrizing its strength. Using the cluster property of local quantum field theory we proved the following relation:

$$\bar{g} = \frac{g}{2}\sqrt{\frac{Z}{2}}$$

valid for general quantum field theories, which yields the expression of $\bar{K}^1$ in terms of g,

$$\bar{K}^1 = \frac{g}{2}.$$

This extends a relation previously conjectured in the case of two-dimensional integrable field theories [6, 7]. In the two-dimensional case, there is no parallel momentum $\vec{k}$ and the rapidity parametrization (2.1) takes the form

$$\omega = m\cosh\vartheta, \qquad \kappa = m\sinh\vartheta. \tag{2.8}$$

[6] Note that this relation remains valid if the Lagrangian field Φ is replaced by any bulk interpolating field for the asymptotic particles and its appropriate wavefunction renormalization Z; in that case $\bar{g}$ also needs to be replaced by another constant which corresponds to the field considered.

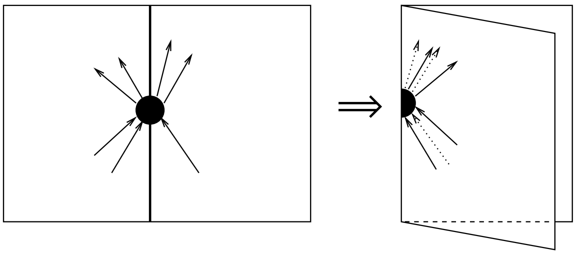

Figure 2. The folding trick, illustrated for a generic defect scattering process.

As a result, the singularity (2.7) corresponds to a pole [1]

$$R(\vartheta) = \frac{\mathrm{i}g^2/2}{\vartheta - \mathrm{i}\pi/2}.$$

Let us now turn to $\bar{K}^2$. Using the reduction formulae derived in [5] the relation to R can be obtained as follows:

$$\bar{K}^2(\kappa, \vec{k}) = R(\omega \to -\mathrm{i}\kappa, \vec{k}).$$

This can be written using the rapidity parametrizations (2.1,2.3) as[7]

$$\bar{K}^2(\vartheta, \vec{k}) = R\left(\mathrm{i}\frac{\pi}{2} + \vartheta, \vec{k}\right)$$

and this relation fits very well with the pictorial representation in figure 1. In two spacetime dimensions this is the same as the relation obtained by Ghoshal and Zamolodchikov[8] in [1]. We remark that when the theory in the bulk is free and the reflection is elastic, the boundary state can be written in a closed form[9]

$$\langle B| = \langle 0| \exp\left\{\bar{K}^1 A_{\mathrm{in}}(0,0) + \int_0^\infty \frac{\mathrm{d}\kappa}{2\pi} \int \frac{\mathrm{d}^{D-1}\vec{k}}{(2\pi)^{D-1}\omega(\kappa, \vec{k})} \bar{K}^2(\kappa, \vec{k}) A_{\mathrm{in}}(-\kappa, -\vec{k}) A_{\mathrm{in}}(\kappa, \vec{k})\right\}. \tag{2.9}$$

3. Defects and defect operators

Boundary conditions considered in the context of the Casimir effect generally allow transmission as well, and such boundaries are called 'defects'. A suitable generalization of the boundary state formalism can be obtained by a folding trick depicted in figure 2, which maps the defect into a boundary system [8]. Suppose now that a defect is located at x_0. In the crossed channel (where time flows in the x direction) it can be represented by a defect operator which acts from the bulk Hilbert space of the $x < x_0$ system into that of the $x > x_0$ system[10].

[7] Note that the rapidity arguments on the two sides of the equality are conceptually different, since they correspond to the kinematical variables of two different channels as defined in (2.1) and (2.3). We can consider them related by analytic continuation.

[8] They also noted that the relation between the two channels can be considered as a generalization of the well-known crossing symmetry to quantum field theories with boundary.

[9] In 1+1 dimensions this can be extended to any integrable QFT with integrable boundary condition [1].

[10] On the two sides of the defect, the bulk theories may differ; in general, a defect can be an interface between very different quantum field theories (as an example one can consider the electromagnetic field in the presence of an interface between two drastically different physical media).

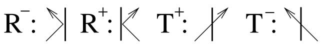

Figure 3. One-particle defect amplitudes.

Let us denote the operator creating the particle for the $x < x_0$ domain as $A_1^\dagger$ while for the $x > x_0$ domain as $A_2^\dagger$. There are now four one-particle reflection amplitudes, shown in figure 3. Two of them are denoted by $R^\pm$ and preserve the species numbers 1, 2, corresponding in the defect picture to reflections on the left and the right side, respectively. The other two, $T^\pm$ are the ones changing 1 into 2 and 2 into 1, and in the defect picture they describe transmission from left to right and right to left, respectively. These can be conveniently put together into a defect matrix[11]

$$D(\vartheta,\vec{k}) = \begin{pmatrix} R^+(\vartheta,\vec{k}) & T^-(\vartheta,\vec{k}) \\ T^+(\vartheta,\vec{k}) & R^-(\vartheta,\vec{k}) \end{pmatrix}.$$

Using the folding map to the boundary system we obtain the defect operator [8] as[12]

$$\begin{aligned} D = 1 + \int_\infty^\infty \frac{d\vartheta}{4\pi} \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \bigg(& R^+\Big(\frac{i\pi}{2} - \vartheta, \vec{k}\Big) A_1^\dagger(-\vartheta,-\vec{k}) A_1^\dagger(\vartheta,\vec{k}) \\ & + T^+\Big(\frac{i\pi}{2} - \vartheta, \vec{k}\Big) A_1^\dagger(-\vartheta,-\vec{k}) A_2(-\vartheta,-\vec{k}) + T^-\Big(\frac{i\pi}{2} - \vartheta, \vec{k}\Big) A_1(\vartheta,\vec{k}) A_2^\dagger(\vartheta,\vec{k}) \\ & + R^-\Big(\frac{i\pi}{2} - \vartheta, \vec{k}\Big) A_2(\vartheta,\vec{k}) A_2(-\vartheta,-\vec{k}) \bigg) + \text{terms with more than two particles.} \end{aligned} \tag{3.1}$$

With the same conditions as for the boundary state (trivial bulk scattering, and elasticity for the combined one-particle reflection/transmission amplitude) the defect operator can be summed up into an exponential form similar to (2.9), as discussed in [4].

4. Derivation of Casimir energy

We now turn to the derivation of Casimir energy of a $(D+1)$-dimensional scalar field $\Phi(t, x, \vec{y})$ in a domain of width L in x (for details see [4, 5]). Consider two defects located at a distance L with defect matrices D_1 and D_2. The ground-state eigenvalue of the Hamiltonian H_B in (2.2) can be evaluated via the partition function. Compactifying all infinite (temporal and spatial) dimensions (i.e. the D extensions perpendicular to x) to circles with perimeter T we can evaluate the partition function in two different ways [4]:

$$Z(L,T) = \mathrm{Tr}_{\mathcal{H}_B}\, e^{-TH_B} = \langle 0|\, D_1\, e^{-LH}\, D_2\, |0\rangle\,,$$

where H is the bulk Hamiltonian (2.4) in the x channel in the domain between the two defects and $|0\rangle$ is the corresponding bulk vacuum state. Inserting a complete set of bulk asymptotic states we obtain

$$Z(L,T) = e^{-LE_0} \sum_n \langle 0|\, D_1\, |n\rangle\, \langle n|\, D_2 |0\rangle\, e^{-L(E_n - E_0)}.$$

Normalizing the bulk ground-state energy E_0 to 0, the first few terms can be written explicitly as

$$1 + \sum_{\vartheta,\vec{k}} \sum_{\vartheta',\vec{k}} \langle 0|D_1|\vartheta,\vec{k};\vartheta',\vec{q}\rangle \langle \vartheta,\vec{k};\vartheta',\vec{q}|D_2|0\rangle\, e^{-L(m_{\mathrm{eff}}(\vec{k})\cosh\vartheta + m_{\mathrm{eff}}(\vec{q})\cosh\vartheta')} + O(e^{-3mL})$$

[11] D is not necessarily unitary, since we allow for inelastic scattering processes creating and annihilating particles.
[12] For the sake of simplicity here we omitted possible one-particle terms corresponding to nontrivial vacuum configurations, but their inclusion using the folding trick is straightforward.

The term 1 is the contribution from the vacuum ($|n\rangle = |0\rangle$), the next term comes from two-particle terms in (3.1) and the higher-order corrections come from the higher multi-particle terms. This is a sort of cluster expansion similar to that used in [7], valid for large values of the volume L. Finite volume restricts the momenta to $\kappa = \frac{2\pi}{T}n$ and $k_i = \frac{2\pi}{T}n_i$, and the normalization of the creation operators becomes

$$[A_{\rm in}(\kappa,\vec{k}), A^{+}_{\rm in}(\kappa',\vec{k}')] = T^D\omega(\kappa,\vec{k})\delta_{\kappa,\kappa'}\delta_{\vec{k},\vec{k}'}.$$

The ground-state (Casimir) energy (per unit transverse area) can be extracted from the partition function as

$$E(L) = -\lim_{T\to\infty}\frac{1}{T^D}\log Z(L,T).$$

The result is

$$E(L) = -\int_{-\infty}^{\infty}\frac{\mathrm{d}\vartheta}{4\pi}\cosh\vartheta\int\frac{\mathrm{d}^{D-1}\vec{k}}{(2\pi)^{D-1}}m_{\rm eff}(\vec{k}) \times R_1^-\left(\frac{\mathrm{i}\pi}{2}+\vartheta,\vec{k}\right)R_2^+\left(\frac{\mathrm{i}\pi}{2}-\vartheta,\vec{k}\right)\mathrm{e}^{-2m_{\rm eff}(\vec{k})L\cosh\vartheta}+\cdots. \tag{4.1}$$

The correction terms correspond to higher particle terms in the expansion (3.1) of the defect operator D and include the amplitudes of reflection/transmission processes involving more than one particle in at least one of the asymptotic states. These can be computed (together with the reflection factors $R^{\pm}$), e.g. using a BQFT formulation as that presented in [3], but it is obvious that they are suppressed by a factor e^{-mL} with respect to the leading order term due to the presence of at least one additional particle in the corresponding term of the expansion of the defect operator D. Note that (4.1) is applicable in the presence of nontrivial bulk and boundary interactions: their effects at leading order are contained in the reflection factors $R^{\pm}$, so as long as there is some theoretical or experimental input from which these can be determined the leading order contribution can be evaluated.

In the elastic case the expansion can be summed up [4][13],

$$E(L) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\vartheta}{4\pi}\cosh\vartheta\int\frac{\mathrm{d}^{D-1}\vec{k}}{(2\pi)^{D-1}}m_{\rm eff}(\vec{k}) \times \log\left(1 - R_1^-\left(\frac{\mathrm{i}\pi}{2}+\vartheta,\vec{k}\right)R_2^+\left(\frac{\mathrm{i}\pi}{2}-\vartheta,\vec{k}\right)\mathrm{e}^{-2m_{\rm eff}(\vec{k})L\cosh\vartheta}\right). \tag{4.2}$$

Let us now calculate the ground-state energy in the presence of nontrivial vacuum configuration of the field. For simplicity we suppose that the boundary is totally reflective. Compactifying the other directions again to circles of perimeter T with periodic boundary conditions we obtain

$$Z(L,T) = \langle B_\alpha|\mathrm{e}^{-LH}|B_\beta\rangle = \sum_n\frac{\langle B_\alpha|n\rangle\langle n|B_\beta\rangle}{\langle n|n\rangle}\mathrm{e}^{-E_nL}.$$

The leading finite-size correction to the ground-state energy for large L is now given by one-particle terms, and the ground-state energy per transverse area (at leading order in L) has the form [5]

$$E_0^{\alpha\beta}(L) = -m\bar{K}^1_\alpha K^1_\beta\,\mathrm{e}^{-mL}+\cdots. \tag{4.3}$$

For partially reflecting boundaries (i.e. defects) the appropriate K^1 coefficient is the one-particle coupling of the defects evaluated in the domain between them. If one of the K^1's is zero then the leading correction comes from two-particle states, and is identical to (4.1).

[13] We remark that the usual zero mode summation method leads to the same result, as indicated in appendix A of [4].

5. Summary and discussion

A very appealing property of the boundary state approach is the universality of the formulae (4.1) and (4.2). In [4] we showed that the latter indeed reproduces all the results previously known for the planar situation, including the famous Lifshitz formula [10] (it also provides a way to compute new cases easily, as we demonstrated for a massive scalar field with Robin boundary condition).

Another important point is that this approach formulates the Casimir effect from an infrared viewpoint. Standard derivations of the Casimir effect solve the microscopic field theory. This necessitates tackling diverse issues such as renormalization, and also the possibility that the infrared (long-distance behaviour) may be quite different from the microscopic description of the theory (as is the case for example in QCD). Formula (4.1) expresses the effect in terms of the asymptotic particles[14], and provides a long-distance expansion for Casimir energy.

Our results are consistent with the philosophy behind the more recent approach by Emig *et al* [11], the origins of which can be found in the earlier papers [12–14]. From this viewpoint the Casimir effect is an interaction of fluctuating surface charge densities, and therefore it does not logically imply the existence of (astronomically large) zero-point energies because the bulk energy density can be trivially discarded. In the boundary state approach the surface is characterized by the coefficients in the cluster expansion of the boundary state (2.5) (or, more generally, the defect operator (3.1)). Both approaches give manifestly finite results, with no ultraviolet divergences whatsoever. There are some differences however. While the boundary state approach only works easily for the planar case, their methods can be used for general geometries. On the other hand, the approach of [11] is only formulated for free field theories with linear boundary conditions since it relies heavily on the computation of Gaussian path integrals, while in the boundary state approach the expansion can be written for interacting field theories with nonlinear boundary conditions, in terms of their long-distance scattering data. The fact that the path integral is Gaussian also gives Emig *et al* the ability to tackle theories with zero mass gap, which is only possible in the boundary state approach whenever the expansion can be resummed into the form (4.2). The boundary state approach, on the other hand, provides access to highly nontrivially interacting theories with a mass gap (a prominent example of which is QCD), provided the relevant scattering data are determined, e.g. from lattice field theory (we remark that it is also highly successful in two-dimensional integrable field theories where exact scattering amplitudes are known).

It is important to note that the restriction of the boundary state approach to the planar case comes from the fact that the high symmetry of the planar situation is exploited to relate the boundary states (or defect operators) to the scattering data, therefore it is not a restriction inherent in any theoretical principle. Finally, we remark that the results (4.1, 4.2) automatically include the contribution of states localized to the defects ('surface plasmons') as discussed in [4].

Note added in proof. After this contribution was accepted we learned of the related results in [15], which are obtained in the case of conformal field theories, whereas our work is concerned with the massive case.

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